A DISCUSSION BY SYNTHETIC METHODS OF THE COVARIANT CONIC OF TWO GIVEN CONICS.*

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At the close of his chapter on Involution, Professor Reye, in his *Geometry of Position*, gives a proof of the following theorem:

Any two tangents to a conic section which pass through two conjugate points of a given involution of points on a line, intersect, in general, upon another fixed conic.

From the fact that conjugate points in an involution are harmonically separated by the double points of that involution, he is able to state the above theorem in the following way:

The pairs of tangents to a conic section which are harmonically separated by two given points intersect, in general, upon another conic.

This theorem, he notes, is a special case of the following, the proof of which is left to the student:

The pairs of tangents to a conic, which are congugate with respect to a second conic, intersect, in general, upon a third conic.

Just how this last theorem is intended to be developed from the theorems that precede is not clear. It is difficult to make any immediate connection between these last two theorems. Owing to the importance of the last theorem, which concerns, indeed, the covariant conic of the two conics, the following discussion is given.

The locus of poles of the tangents to one conic with respect to a second is a third conic, called the polar reciprocal of the first conic with respect to the second; to four harmonic tangents to the first conic correspond four harmonic points on the polar reciprocal conic. These statements follow from the fact that the tangents to the first conic may be considered as the lines joining corresponding points in two projective point rows. These two point rows reciprocate into two projective pencils of rays, corresponding rays of which meet on the polar reciprocal conic. The second part of the theorem follows easily. We have thus a projective correspondence set up between the tangents of one conic and the points of another. We propose now the following problem, which is fundamental for the purpose in hand:

PROBLEM. *Given a pencil of rays of the second order, and a point row of the second order projectively related to it, to find how many of the lines of the pencil pass through the points of the point row that correspond to them.*

Choose a point S on the front row of the second order as the center of a pencil of the first order perspective to it. This pencil will be projective to the pencil of the second order and the locus of the points of intersection of corresponding rays is a cubic curve with a double point at S. (This

*Read before the meeting of the California Section of the American Mathematical Society February 29, 1908.

is proved in the preceding chapter. See also *The Transactions of the American Mathematical Society*, Vol. 3, pp. 372-376, July, 1902.) This cubic will have at most four points in common with the point row of the second order besides the double point S. These points are easily seen to be the points involved in the problem. We see then, that at most four rays of the pencil pass through the points of the point row that correspond to them.

Equipped with this last theorem we are able to discuss the theorem indicated by Professor Reye: *The locus of points of intersection of the tangents of one conic which are conjugate with respect to another is a third conic.*

Given two conics, α and β. From a point, P, which moves along an arbitrary straight line in the plane draw tangents, PA and PA' to the conic α. We wish to find in how many positions of P on the line the two tangents PA and PA' will be conjugate with respect to β. The two systems of tangents PA and PA' are in involution, so that four harmonic tangents, PA, correspond to four harmonic tangents PA'. The pole of PA, the locus of which is the polar reciprocal of α with respect to β, traces out a point row of the second order projective to PA and thus to PA'. At most four of these poles of PA will therefore lie on PA'. The locus is thus a curve of the fourth degree, being cut by an arbitrary line in at most four points. From the theory of poles and polars, however, if PA' pass through the polar of PA, then will PA' pass through the polar of PA, so that the four points in which an arbitrary line meets the locus coincide in pairs; the quartic is thus a pair of coincident conics.

It is clear that the tangents PB and PB', for a point P on this locus, are harmonic conjugates with respect to PA and PA'. The locus of points from which four harmonic tangents may be drawn to two conics is thus a conic. It is in fact the covariant conic of the two conics. For the analytic side of the discussion, see Salmon's *Conic Sections*, pp. 306 and 344. If the anharmonic ratio of the four tangents be different from -1, the quartic found above does not degenerate necessarily, as appears also from the algebraic discussion. The writer does not know of a discussion by synthetic methods of this remarkable conic, which as the above discussion indicates, passes through the eight points of contact of the four common tangents of the two conics.

JOINT MEETING OF MATHEMATICIANS AND ENGINEERS.

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A series of joint meetings of mathematicians and engineers, conducted at The University of Chicago, December 30, 31, 1907, under the auspices of the Chicago Section of the American Mathematical Society, seemed to inaug-